

ON ORTHOGONAL DECOMPOSITION OF $L^2(\Omega)$

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ABSTRACT. *In this short article we show an orthogonal decomposition of the Hilbert space $L^2(\Omega)$ as $L^2(\Omega) = A^2(\Omega) \oplus \frac{d}{dx}(W_0^{1,2}(\Omega))$, define orthogonal projections and see some of its properties. We display some decomposition of elementary functions as corollaries.*

Notations.

Let $\Omega = [0, 1]$

\oplus : Set direct sum

\uplus : Unique direct sum of elements from mutually orthogonal sets

$(\frac{d}{dx})_0^{-2}$: Inverse image of a second order derivative of a traceless function

$A^2(\Omega) = \ker \frac{d}{dx} \cap L^2(\Omega) = \{f : \int_{\Omega} f^2 dx < \infty \ni (\frac{d}{dx})f = 0 \text{ on } \Omega\}$

$\| * \| := \| * \|_{L^2(\Omega)}$



We define the following function spaces

(I) The Hilbert space of square integrable functions over Ω

$$L^2(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}, \text{ measurable and } \int_{\Omega} f^2 dx < \infty\}$$

(II) The Sobolev space

$$W^{1,2}(\Omega) = \{f \in L^2(\Omega) : f'_w \in L^2(\Omega)\}$$

where f'_w is a weak first order derivative of f , i.e.,

$$\exists g \in L_{\text{loc}}(\Omega) : g = f'_w$$

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with

$$\int_{\Omega} g \varphi dx = - \int_{\Omega} f \varphi dx, \forall \varphi \in C_0^{\infty}(\Omega)$$

and

(III) the traceless Sobolev space

$$W_0^{1,2}(\Omega) = \{f \in W^{1,2}(\Omega) : f(0) = f(1) = 0\}$$

The Hilbert space $L_{C^1}^2(\Omega)$ is an inner product space with inner product

$$\langle, \rangle_{L^2(\Omega)} : L^2(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$$

defined by

$$\langle f, g \rangle_{L^2(\Omega)} = \int_{\Omega} f(x) g(x) dx$$

and $W^{1,2}(\Omega)$ with an inner product

$$\langle f, g \rangle_{W^{1,2}(\Omega)} = \left(\langle f, g \rangle_{L^2(\Omega)} + \langle f'_w, g'_w \rangle_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

where f'_w, g'_w are weak first order derivatives.

Definition 1. For

$$f \in L^2(\Omega), \quad \|f\|_{L^2(\Omega)} = \sqrt{\langle f, f \rangle_{L^2(\Omega)}}$$

and for

$$f \in W^{1,2}(\Omega), \quad \|f\|_{W^{1,2}(\Omega)} = \sqrt{\|f\|_{L^2(\Omega)}^2 + \|f'_w\|_{L^2(\Omega)}^2}$$

With respect to the defined inner product above, we have the following orthogonal decomposition

Proposition 1. (*Orthogonal Decomposition*)

$$L^2(\Omega) = A^2(\Omega) \oplus \frac{d}{dx} \left(W_0^{1,2}(\Omega) \right)$$

Proof. We need to show two things.

$$(i) \quad A^2(\Omega) \oplus \frac{d}{dx} \left(W_0^{1,2}(\Omega) \right) = \{0\}$$

$$(ii) \quad \forall f \in L^2(\Omega), \exists g! \in A^2(\Omega) \text{ and } \exists h! \in \frac{d}{dx} \left(W_0^{1,2}(\Omega) \right)$$

such that

$$f = g \uplus h.$$

Indeed

$$(i) \quad \text{Let } f \in A^2(\Omega) \cap \frac{d}{dx} \left(W_0^{1,2}(\Omega) \right).$$

Then

$$f \in A^2(\Omega) \implies \frac{d}{dx}f = 0$$

and so f is a constant. Also

$$f \in \frac{d}{dx} \left(W_0^{1,2}(\Omega) \right)$$

and hence

$$\exists h \in W_0^{1,2}(\Omega)$$

such that

$$f = h'$$

But then as f is a constant and we have

$$h = cx + d$$

But

$$trh = 0 \quad \text{on } \partial\Omega = \{0, 1\}$$

and hence

$$h(0) = 0 \implies d = 0$$

and

$$h(1) = 0 \implies c = 0$$

Therefore

$$h \equiv 0 \quad \text{and hence} \quad f \equiv 0.$$

$$\therefore A^2(\Omega) \cap \frac{d}{dx} \left(W_0^{1,2}(\Omega) \right) = \{0\} \quad (\alpha)$$

(ii) Let $f \in L^2(\Omega)$. Then consider

$$\psi = \left(\frac{d}{dx} \right)_0^{-2} \left(\frac{d}{dx} \right) f$$

which is in $W_0^{1,2}(\Omega)$ and let

$$g = f - \left(\frac{d}{dx} \right) \psi$$

Then

$$\begin{aligned} \frac{d}{dx}g &= \frac{d}{dx} \left(f - \left(\frac{d}{dx} \right) \psi \right) \\ &= \frac{d}{dx}f - \frac{d^2}{dx^2} \left(\left(\frac{d}{dx} \right)_0^{-2} \left(\frac{d}{dx} \right) f \right) \\ &= 0 \end{aligned}$$

Thus

$$g \in A^2(\Omega)$$

and hence with

$$\eta = \left(\frac{d}{dx} \right) \psi \in \frac{d}{dx} \left(W_0^{1,2}(\Omega) \right)$$

we have

$$f = g \uplus \eta \quad (\beta)$$

From (α) and (β) follow the proposition.

Remark. The subset $A^2(\Omega)$ is a closed set and its orthogonal complement

$$\frac{d}{dx} \left(W_0^{1,2}(\Omega) \right) = \left(A^2(\Omega) \right)^\perp$$

as well.

Besides representation of elements of $L^2(\Omega)$ is unique, i.e.,

$$\forall f \in L^2(\Omega), \exists! g \in A^2(\Omega) \quad \text{and} \quad \exists! h \in \frac{d}{dx} \left(W_0^{1,2}(\Omega) \right)$$

such that

$$f = g + h$$

which I denote it as

$$f = g \uplus h$$

Definition 2. Due to the orthogonal decomposition there are two orthogonal projections

$$P : L^2(\Omega) \longrightarrow A^2(\Omega)$$

and

$$Q : L^2(\Omega) \longrightarrow \frac{d}{dx} \left(W_0^{1,2}(\Omega) \right)$$

with

$$Q = I - P$$

where I is the identity operator.

Proposition 2. $\forall f \in L^2(\Omega)$ we have

$$\langle P(f), Q(f) \rangle = 0$$

Proof. Let $f \in L^2(\Omega)$. Then

$$Pf \in A^2(\Omega)$$

and so it is a constant and

$$Qf \in \frac{d}{dx} \left(W_0^{1,2}(\Omega) \right)$$

and hence

$$\exists h \in W_0^{1,2}(\Omega)$$

such that

$$Qf = h' \quad \text{with} \quad \int_\Omega h' dx = 0$$

Therefore

$$\langle P(f), Q(f) \rangle = \langle P(f), h' \rangle = \int_\Omega P(f) h' dx$$

Then from integration by parts we have

$$\int_{\Omega} P(f)h' dx = - \int_{\Omega} P(f)'h dx = 0$$

since $P(f)$ is a constant and we have no boundary integral that might have resulted from the application of integration by parts because of the traceless of h .

$$\therefore \quad \langle P(f), Q(f) \rangle = 0$$

Proposition 3. We have the following properties

$$(i) \quad PQ = 0$$

$$(ii) \quad P^2 = P$$

$$(iii) \quad Q^2 = Q$$

That is P and Q are *idempotent*

Proof. Let $f \in L^2(\Omega)$ and let

$$g = Pf \in A^2(\Omega)$$

Then $g \in L^2(\Omega)$ and let

$$\psi = \left(\frac{d}{dx} \right)_0^{-2} \left(\frac{d}{dx} g \right) = \left(\frac{d}{dx} \right)_0^{-2} (0)$$

Then $\psi = 0$ and setting

$$h = g - \underbrace{\frac{d}{dx} \psi}_0$$

we have

$$g = h + \underbrace{\frac{d}{dx} \psi}_0$$

with

$$Pg = h \quad \text{and} \quad Qg = 0$$

Therefore,

$$Pg = P^2 f = h = g = Pf$$

and

$$Qg = QPf = 0$$

Similarly let

$$\eta = Qf \in \frac{d}{dx} \left(W_0^{1,2}(\Omega) \right)$$

Proof. Let $f \in L^2(\Omega)$. Then we have the unique decomposition,

$$f = Pf + Qf$$

But then

$$\begin{aligned} \langle f, f \rangle &= \langle Pf + Qf, Pf + Qf \rangle \\ &= \langle Pf, Pf \rangle + \langle Qf, Qf \rangle \end{aligned}$$

That is

$$\|f\|^2 = \|Pf\|^2 + \|Qf\|^2$$

We will look at few examples whose validity is supported from *uniqueness* of representations in Hilbert spaces.

Corollary 1.

For $f(x) = x \in L^2(\Omega)$ we have

$$P(f) = \frac{1}{2} \quad \text{and} \quad Q(f) = x - \frac{1}{2}$$

and hence

$$f(x) = \frac{1}{2} \uplus \left(x - \frac{1}{2} \right)$$

Proof. Let

$$\psi = D_0^{-2}(Df) = \left(\frac{d}{dx} \right)_0^{-2} (1) = \frac{1}{2}x^2 - \frac{1}{2}x$$

with

$$\frac{d}{dx}\psi = x - \frac{1}{2}$$

and let

$$g = f - \frac{d}{dx}\psi = \frac{1}{2}$$

Then

$$\frac{d}{dx}(g) = \frac{d}{dx} \left(f - \frac{d}{dx}\psi \right) = 0$$

and hence

$$f = g + \frac{d}{dx}\psi$$

as a direct sum. That is

$$f = \frac{1}{2} \uplus \left(x - \frac{1}{2} \right)$$

Corollary 2. For $f(x) = x$

$$\langle P(f), Q(f) \rangle = 0$$

Proof. Indeed

$$\begin{aligned}\langle P(f), Q(f) \rangle &= \int_{\Omega} \frac{1}{2} \left(x - \frac{1}{2} \right) dx \\ &= \frac{1}{2} \left(\frac{x^2}{2} - \frac{x}{2} \right)_0^1 \\ &= 0\end{aligned}$$

Corollary 3. $\|x\|^2 = \|\frac{1}{2}\|^2 + \|x - \frac{1}{2}\|^2$

Corollary 4. For $f(x) = x^2$

$$P(f) = \frac{1}{3} \quad \text{and} \quad Q(f) = x^2 - \frac{1}{3}$$

Proof. Let

$$\begin{aligned}\psi &= \left(\frac{d}{dx} \right)_0^{-2} \left(\frac{d}{dx} f \right) = \left(\frac{d}{dx} \right)_0^{-2} (2x) \\ \implies \psi(x) &= \frac{1}{3}x^3 - \frac{1}{3}x\end{aligned}$$

and let

$$\begin{aligned}g &= f - \frac{d}{dx}\psi \\ &= x^2 - \left(x^2 - \frac{1}{3} \right) \\ &= \frac{1}{3}\end{aligned}$$

and so

$$g \in \ker \frac{d}{dx}$$

and so

$$f = g \uplus \frac{d}{dx}\psi = \frac{1}{3} \uplus \left(x^2 - \frac{1}{3} \right)$$

which signifies

$$P(f) = \frac{1}{3} \quad \text{and} \quad Q(f) = x^2 - \frac{1}{3}$$

with

$$\left\langle \frac{1}{3}, \left(x^2 - \frac{1}{3} \right) \right\rangle = 0$$

Corollary 5. $\|x^2\|^2 = \|\frac{1}{3}\|^2 + \|(x^2 - \frac{1}{3})\|^2$

Proposition 4. For the orthogonal projections P and Q we have the following results

$$(i) \quad x^n = \frac{1}{n+1} \uplus \left(x^n - \frac{1}{n+1} \right)$$

i.e.

$$P(x^n) = \frac{1}{n+1}, \quad Q(x^n) = x^n - \frac{1}{n+1}$$

$$(ii) \quad e^x = (e-1) \uplus (e^x + 1 - e)$$

i.e.,

$$P(e^x) = e-1, \quad Q(e^x) = e^x + 1 - e$$

$$(iii) \quad P(\cos x) = \sin 1, \quad Q(\cos x) = \cos x - \sin 1$$

so that

$$\cos x = \sin 1 \uplus (\cos x - \sin 1)$$

$$(iv) \quad P(\sin x) = 1 - \cos 1, \quad Q(\sin x) = \sin x + \cos 1 - 1$$

so that

$$\sin x = (1 - \cos 1) \uplus (\sin x + \cos 1 - 1)$$

Proof of (iii). Let

$$\begin{aligned} \psi &= \left(\frac{d}{dx} \right)_0^{-2} \left(\frac{d}{dx} \cos x \right) = \sin x - (\sin 1) x \\ \implies \quad \frac{d}{dx} \psi(x) &= \cos x - \sin 1 \end{aligned}$$

Then set

$$g = f - \frac{d}{dx} \psi = \sin 1 \in \ker \frac{d}{dx}$$

Thus

$$\cos x = \sin 1 \uplus (\cos x - \sin 1)$$

and hence

$$P(\cos x) = \sin 1 \quad \text{and} \quad Q(\cos x) = \cos x - \sin 1$$

Corollary 6.

$$(i) \quad \|x^n\|^2 = \left\| \frac{1}{n+1} \right\|^2 + \left\| x^n - \frac{1}{n+1} \right\|^2$$

$$(ii) \quad \|e^x\|^2 = \|e-1\|^2 + \|e^x + 1 - e\|^2$$

$$(iii) \quad \|\cos x\|^2 = \|\sin 1\|^2 + \|\cos x - \sin 1\|^2$$



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